Homework 4 Solutions

Due: Thursday September 27th at 10:00am in Physics P-124

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1: Let $\mathcal{F}$ be a σ-field on a set Ω and let $f : \Omega' \longrightarrow \Omega$ be any function. Show that

$$f^{-1}(\mathcal{F}) := \{f^{-1}(E) \ : \ E \in \mathcal{F}\}$$

is a σ-field.

Solution:

(1) Since $\Omega \in \mathcal{F}$, we have $\Omega' = f^{-1}(\Omega) \in f^{-1}(\mathcal{F})$.

(2) If $E \in f^{-1}(\mathcal{F})$ then $E = X^{-1}(B)$ for some $B \in f^{-1}(\mathcal{F})$. Since $\Omega - B \in \mathcal{F}$, we have

$$\Omega' - E = \Omega' - f^{-1}(B) = f^{-1}(\Omega) - f^{-1}(B) = f^{-1}(\Omega - B) \in f^{-1}(\mathcal{F}).$$

(3) Finally suppose $(E_n)_{n\in\mathbb{N}}$ is a collection of elements of $f^{-1}(\mathcal{F})$. Then there exists elements $(B_n)_{n\in\mathbb{N}}$ of $\mathcal{F}$ so that $E_n = f^{-1}(B_n)$ for each $n \in \mathbb{N}$. Since $\cup_{n\in\mathbb{N}} B_n \in \mathcal{F}$, we have

$$\cup_{n\in\mathbb{N}} E_n = \cup_{n\in\mathbb{N}} f^{-1}(B_n) = f^{-1}(\cup_{n\in\mathbb{N}} B_n) \in f^{-1}(\mathcal{F}).$$

Hence $f^{-1}(\mathcal{F})$ satisfies the axioms of a σ-field.

Problem 2: Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For measurable functions $f, g : \Omega \longrightarrow \overline{\mathbb{R}}$ on a measure space $(\Omega, \mathcal{F}, \mu)$ show that

(1) $\operatorname{ess\,sup}(f + g) \leq \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g)$ and

(2) $\operatorname{ess\,inf}(f + g) \geq \operatorname{ess\,inf}(f) + \operatorname{ess\,inf}(g)$.

Solution:

(1) For each measurable function $h : \Omega \longrightarrow \overline{\mathbb{R}}$, define

$$U_h := \{z \in \overline{\mathbb{R}} \ : \ \mu(h^{-1}(z, \infty)) = 0\}.$$

If $a \in U_f$ and $b \in U_g$ then

$$(f + g)^{-1}((a + b, \infty)) \supset f^{-1}(a, \infty) \cap g^{-1}(b, \infty)$$

is μ-null and hence $a + b \in U_{f+g}$.
Hence

$$\operatorname{ess\,sup}(f + g) = \inf(U_{f+g}) \leq a + b$$

for all $a \in U_f$ and $b \in U_g$ and so

$$\operatorname{ess\,sup}(f + g) = \inf(U_{f+g}) \leq \inf(U_f) + \inf(U_g) = \operatorname{ess\,sup}(f) + \operatorname{ess\,sup}(g).$$

(2) For each measurable $h : \Omega \longrightarrow \overline{\mathbb{R}}$, we have $h^{-1}((-\infty, a))$ is μ-null if and only if $(-h)^{-1}(-a, \infty)$ is μ-null. Hence

$$\operatorname{ess\,inf}(h) = \sup\{a \in \mathbb{R} \ : \ \mu(h^{-1}((-\infty, a))) = 0\}$$
$$= \sup\{a \in \mathbb{R} \ : \ \mu((-h)^{-1}((-a, \infty))) = 0\}$$
$$= -\inf\{a \in \mathbb{R} \ : \ \mu((-h)^{-1}((a, \infty))) = 0\} = -\operatorname{ess\,sup}(-h).$$

Hence

$$\operatorname{ess\,inf}(f + g) = -\operatorname{ess\,sup}(-f - g) \geq$$
$$-\operatorname{ess\,sup}(-f) - \operatorname{ess\,sup}(-g) = \operatorname{ess\,inf}(f) + \operatorname{ess\,inf}(g).$$

Problem 3: Let $(\Omega, \mathcal{F}, P)$ be a probability space and let X be a random variable. Define the *cumulative distribution function*

$$F_X : \mathbb{R} \longrightarrow \mathbb{R}, \quad F_x(X) := P(X \leq x).$$

(1) Show that there is a countable set $Q \subset \mathbb{R}$ so that $F_X|_{\mathbb{R}-Q}$ is continuous.
(2) Given an example of a random variable whose cumulative distribution function has infinitely many points where it is discontinuous.

Solution:

(1) For each $x \in \mathbb{R}$, define

$$d_x := \lim_{\epsilon \to 0^+} (\inf\{|F_X(z) - F_X(x)| \ : \ z \in (x - \epsilon, x + \epsilon) - \{0\}).$$

We have that F_X is discontinuous at x iff $d_x > 0$. Let $Q_n \subset \mathbb{R}$ be the set of points for which $d_x > \frac{1}{n}$. Let $x_1 < x_2 < \cdots < x_k$ be points in Q_n. Then since F_X is non-decreasing, $F_X(x) \geq \sum_{j=1}^k d_{x_j} \geq \frac{k}{n}$ for all $x > x_k$. Since $F_X(x) \leq 1$, we have that $k \leq n$. Hence Q_n is finite for each $n \in \mathbb{N}$. Hence the set of discontinuous points is equal to

$$\{x \in \mathbb{R} \ : d_x > 0\} = \cup_{n \in \mathbb{N}} Q_n$$

which is a countable union of finite sets and hence is countable.

(2) Consider the probability space $((0, 1], \mathcal{M}|_{(0,1],m})$. Define

$$X : (0, 1] \longrightarrow \mathbb{R}, \quad X(x) = 2^k \text{ if } x \in (2^{-k}, 2^{-k-1}], \text{ for some } k \in \mathbb{N}.$$

Then the image of F_X is countably infinite and hence F_X is discontinuous at infinitely many points.